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Evaluation of General Class of Beta Integrals Involving Certain Particular Functions

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ABSTRACT: In this paper general class of Beta integrals is considered for particular function $\phi(t)$,$\psi(t)$ and f (t) . This class is further used to evaluate certain integrals for some special functions. The class of integrals studied in this paper is integral involving the product of several exponential functions and Gauss's hypergeometric function. As the application of this general integral some integrals for some particular functions are derived.

KEYWORDS: Beta function, Gamma function, Exponential functions, Gauss's hypergeometric function, Extended Beta function, Generalized Hypergeometric function, Hurwitz-Lerch zeta function.

I. INTRODUCTION

The well-known Gamma and Beta function are defined and represented by following integrals. [10]

$$\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt \tag{1.1}$$

And

$$B(\alpha,\beta)=\int_0^\infty t^{\alpha-1}(1-t)^{\beta-1}dt \quad ; \quad \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0 \tag{1.2}$$

Choudhary et. al. [1][2] studied and extended these function to the entire complex plane by inserting the regularization factors $\exp\left(\frac{-A}{t}\right)$ and $\exp\left(\frac{-A}{t(1-t)}\right)$ in the integrands. They studied the following extended form of (1.1) and (1.2) respectively:

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$$\Gamma_A(\alpha)=\int_0^\infty t^{\alpha-1}\exp\left(t-\frac{A}{t}\right)dt;\quad \text{Re}(A)>0 \tag{1.3}$$

$$B(\alpha,\beta;A)=\int_0^\infty t^{\alpha-1}(1-t)^{\beta-1}\exp\left(\frac{-A}{t(1-t)}\right)dt;\quad \text{Re}(A)>0 \tag{1.4}$$

These forms of Beta integral are further unified by the following general integrals. [3]

$$I_{\alpha,\beta}\left[f(t)\right]=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}f(t)dt;\quad \text{Re}(\alpha)>0,\text{Re}(\beta)>0 \tag{1.5}$$

And [5, p.1995, eq.(2.1)]

$$I_{A,\alpha,\beta,\gamma;a,b}\left[\phi(t),\psi(t)\right]=\frac{1}{B(\alpha,\beta)}\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\left[\phi(t)\right]^{\gamma}\exp\left[-A\psi(t)\right]dt;$$

(1.6)

$$\{\text{Re}(\alpha)>0,\text{Re}(\beta)>0,\text{Re}(A)\}>0$$

Where $\phi(t)$, $\psi(t)$ and $f(t)$ are particular functions and $\phi(t)\neq 0,\ a<t<b,\ a\neq b$

Motivated by certain series integral representations of Hurwitz-Zeta function $\zeta(z,a)$ and Hurwitz Lerch zeta function $\phi(z,s,a)$ ([6][7]) very recently. Jaimini and Sharma [4] studied the extended beta function $B(\alpha,\beta;A)$ by the following general series integral representation.

$$B(\alpha,\beta;A)=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp\left[-(A+kp)\psi(t)\right].\left[1-\exp\left(-p\psi(t)\right)\right]^{\lambda}$$

(1.7)

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Where $\psi(t)=\dfrac{1}{t(1-t)}$

On summing the series in (1.7) and using binomial expansion it reduce to the original definition in (1.4).

II. DEFINITIONS AND RESULTS REQUIRED

It is useful here to define the following relations for Pocchammer symbol [10, pp.21-23; eqs. (4), (15), (18),(20),(24)]

$$(a)_n=\begin{cases} a(a+1)...(a+n-1); & if \quad n=1,2,......... \\ 1; & if \quad n=0 \end{cases}$$

$$=\frac{\Gamma(a+n)}{\Gamma(a)} \tag{2.1}$$

$$\Gamma(a-n)=\frac{(-1)^n\Gamma(a)}{(1-a)_n} \tag{2.2}$$

$$(\lambda)_{m+n}=(\lambda)_m(\lambda+m)_n \tag{2.3}$$

The duplication formula defined in the following form.

$$(\lambda)_{2n}=2^{2n}\left(\frac{\lambda}{2}\right)_n\left(\frac{\lambda+1}{2}\right)_n \tag{2.4}$$

The generalized hypergeometric function is defined and represented by the following series [10]

$${}_pF_q\left[\begin{matrix} a_1,...,a_p; \\ \\ b_1,...,b_q; \end{matrix}\; z\right]=\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n n!}, \tag{2.5}$$

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$$p,q \in z^{+}; \quad b_j \neq 0,-1,-2,\ldots; \quad i=1,2,\ldots,r$$

The exponential function ${}_0F_0[-,-;z]$ and the Gauss's hypergeometric function ${}_2F_1[a,b;c;z]$ are special cases of the above hypergeometric function ${}_pF_q[\cdot]$

The following integrals are also required ([9, p.223], see also [11] and [12])

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2}dt = B(\alpha,\beta)F_1[\alpha,\alpha_1,\alpha_2;\alpha+\beta;x_1,x_2]$$

$$(\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0; \quad |x_1|,|x_2|<1) \qquad (2.6)$$

Where F_1 is the Appell's double hypergeometric series defined as follows. [11], [12]

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1-x_1t)^{-\alpha_1}\left(1-x_2(1-t)\right)^{-\alpha_2} dt = B(\alpha,\beta)F_3[\alpha,\beta,\alpha_1,\alpha_2;\alpha+\beta;x_1,x_2]$$

$$(\mathrm{Re}(\alpha)>0, \quad \mathrm{Re}(\beta)>0; \quad |x_1|,|x_2|<1) \qquad (2.7)$$

Where F_3 is Appell's another double hypergeometric series.[11], [12]

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma} \; dt$$

$$= B(\alpha,\beta)(b-a)^{\alpha+\beta-1}(au+v)^{\gamma} \, {}_2F_1\left[\alpha,-\gamma;\alpha+\beta;\frac{-(b-a)u}{(au+v)}\right] \qquad (2.8)$$

$$\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \quad \left|\arg\left(\frac{bu+v}{au+v}\right)\right|<\pi$$

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III. MAIN RESULT

In this section we will study the following general class of integrals in view of the series integral representation of extended Beta function in (1.7)

(i) $$I_{A,\alpha,\beta,\gamma;\lambda_1,p_1,\ldots,\lambda_r,p_r;a,b}\left[\phi(t),\psi(t)\right]$$

$$=\frac{1}{B(\alpha,\beta)}\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\left[\phi(t)\right]^{\gamma}\exp\left[-A\psi(t)\right]\prod_{i=1}^{r}\left[1-\exp\left(-p_i\psi(t)\right)\right]^{\lambda_i}dt$$

(3.1)

$\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(A)>0 \quad \lambda_i, p_i>0$ for $i=1,2,\ldots,r$

EVALUATION OF INTEGRALS: -

The following three integrals are evaluated for certain values of the particular functions $\phi(t),\psi(t)$ and $f(t)$ in our main general class of integrals considered in (3.1)

Integral-1

For $\phi(t)=(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2} \quad \psi(t)=\frac{1}{t(1-t)},\ a=0 \text{ and } b=\gamma=1$

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We have

$$I_{A,\alpha,\beta,1;\lambda_1,p_1,\ldots,\lambda_r,p_r;0,1}\left[\left(1-x_1t\right)^{-\alpha_1}\left(1-x_2t\right)^{-\alpha_2},\frac{1}{t\left(1-t\right)}\right]$$

$$=\sum_{m_1,m_2,n_1,\ldots,n_r=0}^{\infty}\frac{\prod_{i=1}^{r}\left[\left(-\lambda_i\right)_{n_i}\right]\left(\alpha_1\right)_{m_1}\left(\alpha_2\right)_{m_2}\left(\alpha\right)_{m_1+m_2}x_1^{m_1}x_2^{m_2}}{n_1!\ldots n_r!\ \left(\alpha+\beta\right)_{m_1+m_2}m_1!\,m_2!}$$

$${}_2F_2\left[\begin{matrix}\dfrac{\left(1-\alpha-\beta-m_1-m_2\right)}{2},\dfrac{\left(2-\alpha-\beta-m_1-m_2\right)}{2}\\ \left(1-\beta\right)\qquad ,\left(1-\alpha-\beta-m_1-m_2\right)\end{matrix};-4\left(A+\sum_{i=1}^{r}p_in_i\right)\right]$$

(3.2)

Where $\operatorname{Re}(\alpha),\operatorname{Re}(\beta)>0$ $|x_1|,|x_2|<1$ and the result in (3.2) exists.

Integral-2

For $\phi(t)=\left(1-x_1t\right)^{-\alpha_1}\left(1-x_2\left(1-t\right)\right)^{-\alpha_2}$ $\psi(t)=-\frac{1}{t\left(1-t\right)}$, $a=0$ and $b=\gamma=1$

We have

$$I_{A,\alpha,\beta,1;\lambda_1,p_1,\ldots,\lambda_r,p_r;0,1}\left[\left(1-x_1t\right)^{-\alpha_1}\left(1-x_2\left(1-t\right)\right)^{-\alpha_2},-\frac{1}{t\left(1-t\right)}\right]$$

$$=\sum_{m_1,m_2,n_1,\ldots n_r=0}^{\infty}\frac{\prod_{i=1}^{r}\left[\left(-\lambda_i\right)_{n_i}\right]\left(\alpha_1\right)_{m_1}\left(\alpha_2\right)_{m_2}x_1^{m_1}x_2^{m_2}\left(\alpha\right)_{m_1}\left(\beta\right)_{m_2}}{n_1!\ldots n_r!\qquad\left(\alpha+\beta\right)_{m_1+m_2}m_1!m_2!}.$$

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$$
{}_2F_2\left[\begin{matrix}\frac{(1-\alpha-\beta-m_1-m_2)}{2},\frac{(2-\alpha-\beta-m_1-m_2)}{2}\\(1-\alpha-m_1),(1-\beta-m_2)\end{matrix};4\left(A+\sum_{i=1}^{r}p_in_i\right)\right] \tag{3.3}
$$

Where $\mathrm{Re}(\alpha),\ \mathrm{Re}(\beta)>0,\ |x_1|,|x_2|<1$ and the result in (3.3) exists.

Integral-3

For $\phi(t)=(ut+v)$ and $\psi(t)=\frac{1}{(t-a)(b-t)}$

We have $I_{A,\alpha,\beta,\gamma;\lambda_1,p_1,\ldots,\lambda_r,p_r;a,b}\left[(ut+v),\frac{1}{(t-a)(b-t)}\right]$

$$
=(b-a)^{\alpha+\beta-1}(au+v)^{\gamma}\sum_{m_1,n_1,\ldots,n_r=0}^{\infty}\frac{\prod_{i=1}^{r}[(-\lambda_i)n_i](\alpha)_{m_1}(-\gamma)_{m_1}}{n_1!\ldots n_r!\ (\alpha+\beta)_{m_1}\ m_1!}\left[-\frac{(b-a)u}{(au+v)}\right]^{m_1}.
$$

$$
{}_2F_2\left[\begin{matrix}\frac{(1-\alpha-\beta-m_1)}{2},\frac{(2-\alpha-\beta-m_1)}{2}\\(1-\beta)\quad,\ (1-\alpha-m_1)\end{matrix};\ \frac{-4\left(A+\sum_{i=1}^{r}p_in_i\right)}{(b-a)^2}\right] \tag{3.4}
$$

Where $\mathrm{Re}(\alpha),\ \mathrm{Re}(\beta)>0,\ \left|\arg\left(\frac{bu+v}{au+v}\right)\right|<\pi,\quad a\neq b$ and the result in (3.4) exists.

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OUT LINE OF PROOFS:

Proof of (3.2)

To prove the result in (3.2) we denote its left-hand side by Δ_1 i.e.

$$\Delta_1 = I_{A,\alpha,\beta,\lambda_1,p_1\ldots\lambda_r,p_r;0,1}\left[(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2},\frac{1}{t(1-t)}\right]$$

Now in view of the general notation of class of integrals (3.1) we have the following form of integral.

$$\Delta_1 = \frac{1}{B(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2}.$$

$$\exp\left[-\frac{A}{t(1-t)}\right]\prod_{i=1}^{r}\left[1-\exp\left(\frac{-p_i}{t(1-t)}\right)\right]^{\lambda_i} dt$$

Now using binomial expansion and then expressing exponential function in series form ${}_0F_0$ with the help of (2.5) and then changing the order of summation and integration we have

$$\Delta_1 = \frac{1}{B(\alpha,\beta)}\sum_{m,n_1,\ldots,n_r=0}^{\infty}\frac{\prod_{i=1}^{r}\left[(-\lambda_i)_{n_i}\right]}{n_1!\ldots n_r!}\frac{\left(-A-\sum_{i=1}^{r}p_in_i\right)^m}{m!}.$$

$$\int_0^1 t^{(\alpha-m)-1}(1-t)^{(\beta-m)-1}(1-x_1t)^{-\alpha_1}(1-x_2t)^{-\alpha_2}\,dt$$

Now on evaluating the inner integral with the help of (2.6) and using their in the definition of Appell function F_1 and then interpreting the m series into ${}_2F_2$ in view of (2.1) to (2.4) and (2.5) we atonce arrived at the desired result in (3.2)

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The Results (3.3) and (3.4) are proved following on similar lines as to prove in (3.2) and using (2.7) (2.8) there in respectively.

SPECIAL CASES:

If in (3.2), (3.3) and (3.4), we take $r = 1$ these reduce to the known results due to Pathan, Jaimini and Sharma [9, pp.5-8, Eqs.3.3-3.5] respectively.

If in main result (3.2), (3.3) and (3.4) we take $\lambda = 0$ and $r = 1$ then these reduce immediately to the known results due to khan et.al [5, pp.1995-1996, eqs. (2.2), (2.4), (2.6)] respectively.

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