International Journal Of Multidisciplinary Research In Science, Engineering and Technology (IJMRSET)

| ISSN: 2582-7219 | www.ijmrset.com| Impact Factor: 4.988|Monthly, Peer Reviewed & Referred Journal |

| Volume 2, Issue 9, September 2019 |

BESSEL FUNCTIONS IN MATHEMATICS

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ABSTRACT: Bessel function, also called cylinder function, any of a set of mathematical functions systematically derived around 1817 by the German astronomer Friedrich Wilhelm Bessel during an investigation of solutions of one of Kepler's equations of planetary motion. Particular functions of the set had been formulated earlier by the Swiss mathematicians Daniel Bernoulli, who studied the oscillations of a chain suspended by one end, and Leonhard Euler, who analyzed the vibrations of a stretched membrane.

After Bessel published his findings, other scientists found that the functions appeared in mathematical descriptions of many physical phenomena, including the flow of heat or electricity in a solid cylinder, the propagation of electromagnetic waves along wires, the diffraction of light, the motions of fluids, and the deformations of elastic bodies. One of these investigators, Lord Rayleigh, also placed the Bessel functions in a larger context by showing that they arise in the solution of Laplace's equation when the latter is formulated in cylindrical (rather than Cartesian or spherical) coordinates.

KEYWORDS-bessel,functions,mathematics,cylinder,solid,electromagnetic,diffraction

I. INTRODUCTION

Specifically, a Bessel function is a solution of the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0,$ which is called Bessel's equation. For integral values of n, the Bessel functions

$$J_n(x) = \frac{x^n}{2^n n!}\left[1-\frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right].$$

are

The graph of $J_0(x)$ looks like that of a damped cosine curve, and that of $J_1(x)$ looks like that of a damped sine curve (see graph).

Certain physical problems lead to differential equations analogous to Bessel's equation; their solutions take the form of combinations of Bessel functions and are called Bessel functions of the second or third kind.

BESSEL FUNCTIONS ARISE IN MANY PROBLEMS in physics possessing cylindrical symmetry, such as the vibrations of circular drumheads and the radial modes in optical fibers. They also provide us with another orthogonal set of basis functions.

Bessel functions have a long history and were named after Friedrich Wilhelm Bessel (1784−18461784−1846)[1,2,3]

The first occurrence of Bessel functions (zeroth order) was in the work of Daniel Bernoulli on heavy chains (1738). More general Bessel functions. were studied by Leonhard Euler in 1781 and in his study of the vibrating membrane in 1764.1764. Joseph Fourier found them in the study of heat conduction in solid cylinders and Siméon Poisson (1781-1840) in heat conduction of spheres (1823).

The history of Bessel functions, did not just originate in the study of the wave and heat equations. These solutions originally came up in the study of the Kepler problem, describing planetary motion. According to G.NG.N. Watson in his Treatise on Bessel Functions, the formulation and solution of Kepler's Problem was discovered by Joseph-Louis Lagrange (1736-1813), in 1770. Namely, the problem was to express the radial coordinate and what is called the eccentric anomaly, E, as functions of time. Lagrange found expressions for the coefficients in the expansions of r and E in trigonometric functions of time. However, he only computed the first few coefficients. In 1816, Friedrich Wilhelm Bessel (1784−1846)(1784−1846) had shown that the coefficients in the expansion for r could be given an integral representation. In 1824 , he presented a thorough study of these functions, which are now called Bessel functions.

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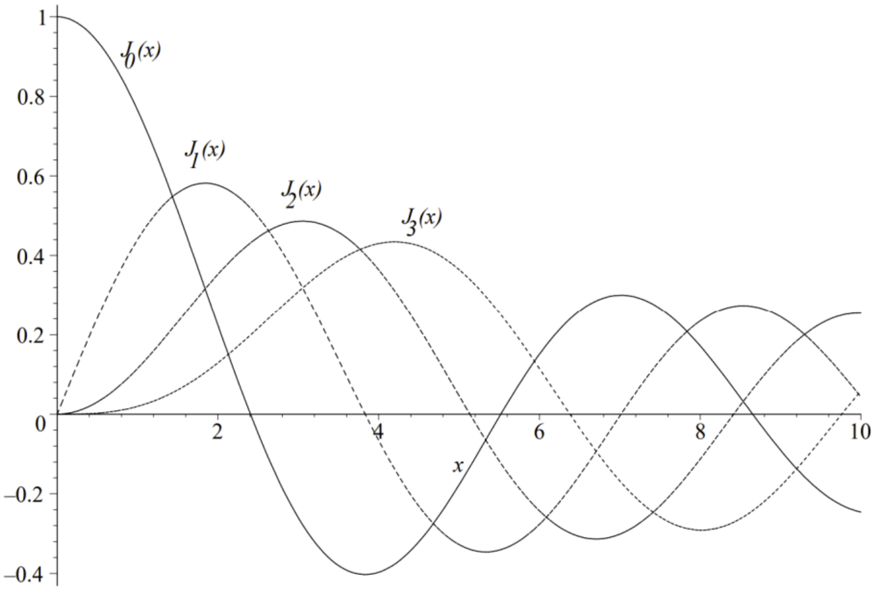
II. DISCUSSION

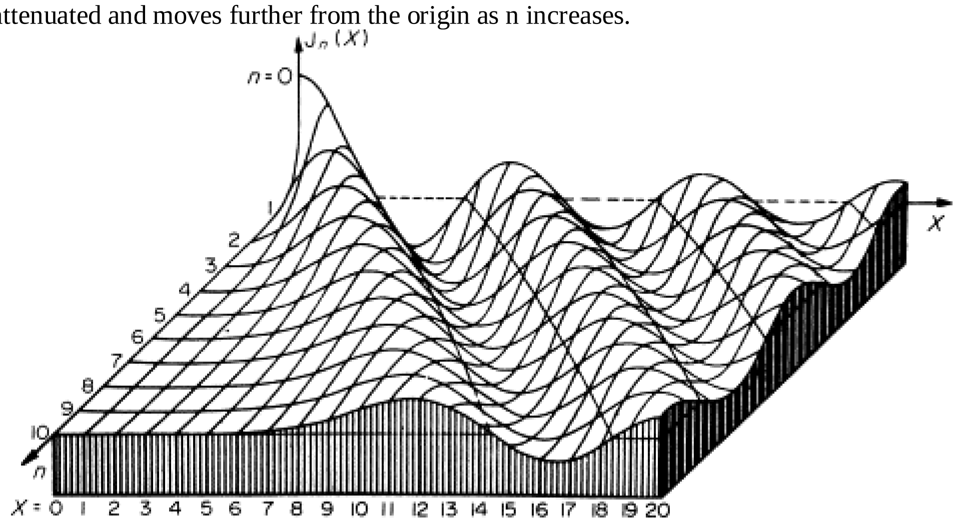


Plots of the Bessel functions

In many applications, one desires bounded solutions at x=0.These functions do not satisfy this boundary condition. For example, one standard problem is to describe the oscillations of a circular drumhead. For this problem one solves the two dimensional wave equation using separation of variables in cylindrical coordinates. The radial equation leads to a Bessel equation. The Bessel function solutions describe the radial part of the solution and one does not expect a singular solution at the center of the drum. The amplitude of the oscillation must remain finite. Thus, only Bessel functions of the first kind can be used.[5,7,8]

Bessel functions satisfy a variety of properties, which we will only list at this time for Bessel functions of the first kind. The reader will have the opportunity to prove these for homework.

The shapes of Bessel functions up to order ten are illustrated in Figure 16. It may be seen that $J_0(X) = 1$ at the origin and fluctuates in a decreasing manner as X increases. All the other $J_n(X)$ are zero at the origin and the first peak is progressively attenuated and moves further from the origin as n increases.



III. RESULTS

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$$j_0(x) = \frac{\sin x}{x}.$$
$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},$$
$$j_2(x) = \left(\frac{3}{x^2} - 1\right)\frac{\sin x}{x} - \frac{3\cos x}{x^2},$$
$$j_3(x) = \left(\frac{15}{x^3} - \frac{6}{x}\right)\frac{\sin x}{x} - \left(\frac{15}{x^2} - 1\right)\frac{\cos x}{x}$$

Bessel's function

$$J_{-\left(\frac{3}{2}\right)}(x) = \sqrt{\frac{2}{\pi x}}\left(-sinx - \frac{cosx}{x}\right)$$

Bessel functions are mathematical solutions to the Wave Equation in cylindrical geometry. They behave kind of like you would expect them to.

(1) Throw a rock in a lake. Notice the ring of waves that expand away from where the rock struck the water. Notice that the outer most part of the wave ring has a longer wavelength than the innermost part of the ring of waves. Notice that the amplitude of the waves decreases as you get further from where the rock struck the water. If you are very careful, you'll notice that the amplitude of the wave ring scales like the the inverse square root of the distance from where the rock struck. This makes perfect sense, physically, just from conservation of energy.

(2) If you want to have a little bit of fun, get a snare drum and toss a little bit of sand on it. Then get a nice loud stereo and play some nice loud sine wave tones through the speaker. Change the frequency slowly, and watch the sand form interesting patterns on the drumskin. That is you looking at the Bessel function Eigen modes of that drumskin under tension, whose amplitude variations are described by a wave equation, and whose solution, with that circular boundary, are Bessel Functions.[8,9]

One of the biggest problems that students have in understanding Bessel Functions is when they have instructors tell them that Bessel Functions are mysterious solutions to a differential equation. No, Bessel Functions are not mysterious at all. They're just sines and cosines in cylindrical geometry.

In 3D the solutions are the Spherical Bessel Functions.

IV. CONCLUSIONS

Bessel functions (named after the astronomer F.W. Bessel) are solutions to differential equations:

x^2y&prime&prime + xy′ + (x^2 – y^2)y = 0

Where:

- n is a non-negative real number.

Function values don't usually have to be calculated by hand; They can be found in many tables (like these Bessel tables).

The solutions are called Bessel functions of order n or—less commonly—cylindrical functions of order n. They are one of the most widely used functions in applied mathematics and are popular in problems that involve circular or cylindrical symmetry, so are sometimes called cylinder functions. They are also important in the study of wave propagation.

Solutions to Bessel's Equation

Bessel's equation is a second-order differential equation with two linearly independent solutions:

- Bessel function of the first kind,
- Bessel function of the second kind.

Bessel Function of the first kind[**6**,7]

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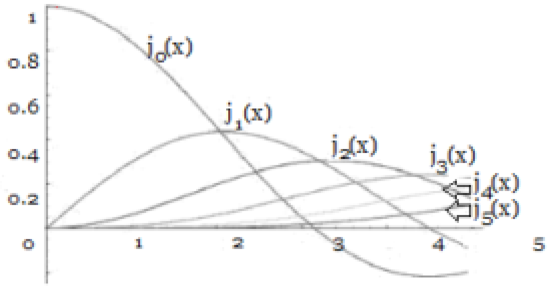

Bessel functions for various orders.

Bessel functions of the first kind (sometimes called ordinary Bessel functions), are denoted by $J_n(x)$, where n is the order.

$$J_v(z) = \left(\tfrac{1}{2}z\right)^v \sum_{k=0}^{\infty} \frac{\left(-\tfrac{1}{4}z^2\right)^k}{k!\,\Gamma(v+k+1)}$$

Bessel Function of the second kind

The second solution (Y_v or N_v) is called a Bessel Function of the second kind and is denoted by $n_n(x)$. It can also be expressed as a Neumann function:

$$Y_v(z) = \frac{J_v(z)\cos(v\pi) - J_{-v}(z)}{\sin(v\pi)}$$

Uses[**4**,5,**6**]

A large number of fields use Bessel functions, including:

- Acoustic theory,
- Electric field theory,
- Hydrodynamics,
- Nuclear Physics,
- Radio Physics.[7,8,9]

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